A cohomological interpretation of Brion's formula

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A subset K of $\mathbb{R}^n$ gives rise to a formal LAURENT series with monomials corresponding to lattice points in K. Under suitable hypotheses, this series represents a rational function $R(K)$. MICHEL BRION has discovered a surprising formula [Bri88] relating the rational function $R(P)$ of a lattice polytope P to the sum of rational functions corresponding to the supporting cones subtended at the vertices of P. The result is re-phrased and generalised in the language of cohomology of line bundles on complete toric varieties. BRION's formula is the special case of an ample line bundle on a projective toric variety.—The paper also contains some general remarks on the cohomology of torus-equivariant line bundles on complete toric varieties, valid over arbitrary commutative ground rings.

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1 Introduction

The main result of this paper is a generalisation of a formula discovered by BRION, relating the lattice point enumerator of a rational polytope to the lattice points enumerators of supporting cones subtended at its vertices [Bri88, §2.2] (see [BHS] for an introduction to the theory, and [Hütb] for an elementary geometric proof). In spirit the proof of the generalisation is similar to BRION's original exposition, but avoids the use of equivariant K-theory in favour of a more elementary treatment of cohomology of line bundles on complete toric varieties.

Since line bundles are encoded by support functions defined on a fan, the result can be re-formulated in combinatorial terms. This has been done for upper convex support functions (corresponding to line bundles which are generated by global sections) by ISHIDA [Ish90, Theorem 2.3], generalising the original result of BRION. The present paper goes one step further and includes the case of arbitrary, non-convex support functions.

We will give a precise formulation of the result below. Roughly speaking, we prove that a sum of certain rational functions, all given by infinite LAURENT series, degenerates to a LAURENT polynomial, and interpret the coefficients of the

occurring monomials as homogeneous EULER characteristics of the sheaf cohomology of an algebraic line bundle.

The proof relies on a non-standard computation of the cohomology of line bundles on complete toric varieties (Theorem 2.7) which is similar to, but slightly easier than, the standard result as given by ODA [Oda88, Theorem 2.6]. This computation in turn depends on a variant of ČECH cohomology (Proposition 2.2) which should be well-known; since it seems not to be well-documented in available publications, we include a proof at the end of the paper.

Notational conventions and the main result

We have to introduce some notation first. Let $M \cong \mathbb{Z}^n$ be a lattice of rank n. We call the set of maps $S = \mathrm{map}(M, \mathbb{C})$ the set of *formal* LAURENT *series*. Given an element $\mathbf{b} \in M$ we let $x^{\mathbf{b}} \in S$ denote the map which is zero on $M \setminus \{\mathbf{b}\}$, and takes the value 1 on $\mathbf{b}$. We call $x^{\mathbf{b}}$ the LAURENT *monomial* with exponent $\mathbf{b}$.

The terminology can be justified. Given a choice of basis $e_1, e_2, \ldots, e_n$ of M we can write every element $\mathbf{b} \in M$ uniquely as $\mathbf{b} = \sum_j b_j e_j$ with $b_j \in \mathbb{Z}$. Then for $f \in S$ the formal sum

$$\sum_{\mathbf{b}\in M} f(\mathbf{b}) \cdot x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n}$$

is a LAURENT series in the indeterminates $x_1, x_2, \ldots, x_n$. The map $x^{\mathbf{b}}$ corresponds to the product $x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n}$, *i.e.*, a series with a single non-trivial summand.

Let $P \subset S$ denote the subset of maps with finite support; in particular, it contains the maps $x^{\mathbf{b}}$ defined above. After choosing a basis of M we can identify P with the ring of LAURENT polynomials in n indeterminates; on the level of maps, the product is given by a convolution formula. The same formula equips S with the usual structure of a P-module.

Set $M_{\mathbb{R}} = M \otimes \mathbb{R} \cong \mathbb{R}^n$. We consider M as a subset of $M_{\mathbb{R}}$ using the natural identification $M = M \otimes 1$. Given a subset $K \subseteq M_{\mathbb{R}}$ and an element $b \in M_{\mathbb{R}}$ we define

$$b + K = \{b + x \,|\, x \in K\} \qquad \text{and} \qquad -K = \{-x \,|\, x \in K\} \ .$$

1.1 Definition. For a subset $K \subseteq M_{\mathbb{R}}$ we define the formal LAURENT series

$$R[K] = \sum_{\mathbf{a}\in M\cap K} x^{\mathbf{a}} \ \in S \ .$$

A straightforward calculation shows $R[\mathbf{b} + K] = x^{\mathbf{b}} R[K]$ for any $\mathbf{b} \in M$.

In favourable cases, for example when K is a pointed rational polyhedral cone in $M_{\mathbb{R}}$, the series $R[K]$ represents a rational function (an element in the quotient field $Q(P)$ of P) which we will denote $R(K) \in Q(P)$.

As an explicit example, for $K = \mathbb{R}_{\leq 2} = 2 + \mathbb{R}_{\leq 0} \subset \mathbb{R}$, we have $R[K] = x^2 R[\mathbb{R}_{\leq 0}] = x^2 \sum_{a\leq 0} x^a$, so $R(K) = x^2/(1 - x^{-1})$. See [BHS] for more examples.

Let $N = \hom_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathbb{Z}^n$ be the dual lattice of M. Then $N_{\mathbb{R}} = N \otimes \mathbb{R} \cong \mathbb{R}^n$ is naturally the dual of the $\mathbb{R}$-vector space $M_{\mathbb{R}}$. The duals of N and $N_{\mathbb{R}}$ are canonically isomorphic to M and $M_{\mathbb{R}}$, respectively.

Let Σ be a finite complete fan in $N_{\mathbb{R}}$, consisting of strongly convex rational polyhedral cones, and denote by X_Σ the associated toric variety defined over $\mathbb{C}$. (See [Oda88] for details on cones, fans, and the relation to varieties.) Let $h\colon N_{\mathbb{R}} \longrightarrow \mathbb{R}$ be a support function on Σ; on each cone $\sigma \in \Sigma$ it coincides with a linear function $h_\sigma \otimes \mathrm{id}_{\mathbb{R}}$ for some $h_\sigma \in \hom_{\mathbb{Z}}(N, \mathbb{Z}) = M$. Define the rational function

$$R(\Sigma, h) = \sum_{\substack{\sigma \in \Sigma \\ \dim \sigma = n}} R(-h_\sigma + \sigma^\vee)$$

where $\sigma^\vee = \{x \in M_{\mathbb{R}} \mid \forall y \in \sigma : \langle x, y\rangle \geq 0\}$ is the dual cone, defined using the standard evaluation pairing $\langle x, y\rangle = y(x)$. Denote the (algebraic) line bundle on X_Σ associated to h by L_h; see 2.4 below for an explicit description. Given a vector $\mathbf{a} \in M$ we write $H^k(X_\Sigma; L_h)_{\mathbf{a}}$ for the homogeneous part of degree $\mathbf{a}$ of the kth sheaf cohomology of L_h; see 2.5–2.6 below for an elementary description.

1.2 Theorem. *The rational function $R(\Sigma, h)$ is a* Laurent *polynomial. The coefficient of the monomial $x^{\mathbf{a}}$ in the polynomial $R(\Sigma, h)$ is the* Euler *characteristic of $H^*(X_\Sigma; L_h)_{\mathbf{a}}$, so it is given by the alternating sum*

$$\chi(L_h)_{\mathbf{a}} = \sum_{k=0}^{n} (-1)^k \dim_{\mathbb{C}} H^k(X_\Sigma; L_h)_{\mathbf{a}} \; .$$

In short, we have the equality

$$R(\Sigma, h) = \sum_{\mathbf{a} \in M} \chi(L_h)_{\mathbf{a}} \cdot x^{\mathbf{a}} \; . \tag{1}$$

Brion's formula for lattice polytopes

For an n-dimensional polytope $K \subset M_{\mathbb{R}}$ with vertices in M, let Σ_K denote the inner normal fan of K. The support function of K, given by

$$h_K\colon N_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad x \mapsto -\inf\{\langle p, x\rangle \mid p \in K\} \; ,$$

defines a support function on Σ_K. Explicit calculation of sheaf cohomology shows

$$H^k(X_{\Sigma_K}; L_{h_K}) = \begin{cases} 0 & \text{if } k \neq 0 \\ \bigoplus_{M \cap K} \mathbb{C} & \text{if } k = 0 \end{cases}$$

(see [Oda88, Corollary 2.9], or [Hüta, Theorem 2.5.3] for an elementary proof). If σ is an n-dimensional cone in Σ_K, and h_σ is the linear function associated to h_K and σ, then $-h_\sigma + \sigma^\vee$ is the support cone of K subtended at the vertex corresponding to σ. Thus Theorem 1.2 reduces to the original theorem of Brion [Bri88, §2.2]: The rational function $R(\Sigma_K, h_K)$ is a Laurent polynomial with terms corresponding to the integral points of K.

Similarly, by considering the support function $-h$, and using the calculation

$$H^k(X_{\Sigma_K}; L_{-h_K}) = \begin{cases} 0 & \text{if } k \neq n \\ \bigoplus_{M \cap \operatorname{int}(-K)} \mathbb{C} & \text{if } k = n \end{cases}$$

(see [Oda88, Corollary 2.8], or [Hüta, Theorem 2.5.3] for an elementary proof), we see that the rational function $R(\Sigma_K, -h_K)$ is a LAURENT polynomial with summands corresponding to the integral points in the interior of $-K$, up to a factor of $(-1)^n$ [Bri88, §2.5].

Finally, by considering a globally linear support function $h = \mathbf{a} \in M$, so that $L_{\mathbf{a}} \cong \mathcal{O}_{X_{\Sigma_K}}$, Theorem 1.2 together with the calculation

$$H^k(X_{\Sigma_K}; L_{\mathbf{a}}) = \begin{cases} 0 & \text{if } k \neq 0 \\ \mathbb{C} & \text{if } k = 0 \end{cases}$$

with H^0 concentrated in homogeneous degree $\mathbf{a}$ (see [Hüta, Theorem 2.5.3] for the case $\mathbf{a} = 0$, the general case follows easily) yield the identity $R(\Sigma_K, \mathbf{a}) = x^{\mathbf{a}}$ (cf. [Ish90, Corollary 2.4]; see [BV97, Proposition 3.1] for a LAURENT series version).

The cohomology calculations in this subsection can be done with the aid of Theorem 2.7 below; in essence, one has to check that certain subcomplexes of the sphere S^{n-1} are contractible. This is what is behind the calculations in the paper [Hüta] which, however, uses a dual point of view, using the fact that the fans considered above are normal fans of polytopes. We omit the details.

Ishida's formula

If the support function h is upper convex (equivalently, if the associated line bundle is generated by global sections), the negatives of the linear functions h_σ for n-dimensional cones $\sigma \in \Sigma$ span a polytope Q in $M_{\mathbb{R}}$ with vertices in M. Since

$$H^k(X_{\Sigma_P}; L_h) = \begin{cases} 0 & \text{if } k \neq 0 \,, \\ \bigoplus_{M \cap Q} \mathbb{C} & \text{if } k = 0 \,, \end{cases}$$

Theorem 1.2 specialises to [Ish90, Theorem 2.3] for complete fans.

An explicit example

1.3 Example. We consider the case $n = 2$, $N = \mathbb{Z}^2$ and $N_{\mathbb{R}} = \mathbb{R}^2$. Let Σ be the unique complete fan in $N_{\mathbb{R}}$ whose 1-cones are generated by the following four vectors:

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad v_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Let $X, Y \in M = \hom_{\mathbb{Z}}(N, \mathbb{Z})$ denote the dual of the standard basis of $N = \mathbb{Z}^2$. Let $h \colon N_{\mathbb{R}} \longrightarrow \mathbb{R}$ be the support function specified by the values

$$h(v_1) = 0 \qquad h(v_2) = -2 \qquad h(v_3) = 0 \qquad h(v_4) = -2$$

given by extending linearly over cones; for example, on $\sigma = \mathrm{cone}(v_1, v_2)$ it agrees with the linear function $h_\sigma = 2X - 2Y \in M$ which corresponds to a LAURENT monomial written x^2y^{-2} . Using Theorem 2.7 we can explicitly compute the cohomology of L_h (see end of §2 below). It turns out that $H^0(X_\Sigma; L_h) = 0$, and that $\dim H^2(X_\Sigma; L_h) = 1$ concentrated in homogeneous degree $-Y \in M$. The space $H^1(X_\Sigma; L_h)$ is 4-dimensional, with a 1-dimensional contribution coming from degrees 0, $-X+Y$, Y and $X+Y$. The right-hand side of Equation (1) thus is (denoting the indeterminates again by x and y)

$$-1 - x^{-1}y - y - xy + y^{-1} \ .$$

The left-hand side is worked out easily as well. For example, the summand corresponding to $\sigma = \mathrm{cone}(v_1, v_2)$ is the rational function represented by the lattice point enumerator of the shifted cone

$$-h_\sigma + \sigma^\vee = (-2X + 2Y) + \mathrm{cone}\,(\,X, -X+Y\,) \ \subset M_\mathbb{R}$$

or, in coordinates of $M_\mathbb{R} \cong \mathbb{R}^2$,

$$-h_\sigma + \sigma^\vee = \begin{pmatrix} -2 \\ 2 \end{pmatrix} + \mathrm{cone}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) \ .$$

This lattice point enumerator is given by

$$\frac{x^{-2}y^2}{(1 - x^{-1}y)(1-x)} \ .$$

In total, the left-hand side of Equation (1) equals

$$\frac{x^{-2}y^2}{(1 - x^{-1}y)(1-x)} + \frac{x^2y^2}{(1 - x^{-1})(1-xy)}$$
$$+ \frac{x^{-2}y^{-2}}{(1 - x^{-1})(1-x^{-1}y^{-1})} + \frac{x^2y^{-2}}{(1 - x)(1-xy^{-1})}$$

which coincides with the LAURENT polynomial $-1 - x^{-1}y - y - xy + y^{-1}$, as an explicit calculation shows.

2 Cohomology of line bundles

Čech cohomology of quasi-coherent sheaves

2.1 Let Σ be a finite complete fan in $N_\mathbb{R}$. By taking intersection of positive-dimensional cones with the unit sphere S^{n-1} (defined with respect to any inner product) the fan induces the structure of a regular CW complex on S^{n-1}. Given a cone $\sigma \in \Sigma$ we write $\bar\sigma = \sigma \cap S^{n-1}$ for the corresponding cell of S^{n-1}; this includes the case of the empty cell $\bar 0$. We fix once and for all orientations of the cells and write $[\bar\sigma : \bar\tau]$ for the incidence number of $\bar\sigma$ and $\bar\tau$. By convention,

we have $[\bar\tau : \bar 0] = 1$ for all 1-dimensional cones $\tau \in \Sigma$. Regularity of the *CW* decomposition implies that $[\bar\sigma : \bar\tau] \in \{-1, 0, 1\}$ for all cones $\sigma, \tau \in \Sigma$. Note that in the (augmented) cellular chain complex of S^{n-1} the empty cell corresponds to the augmentation (concentrated in degree -1).

The computation of ČECH cohomology does not depend on using the complex numbers as a ground field. So let A denote a commutative ring, and let X_Σ denote the toric A-scheme associated to Σ; it is obtained by gluing the affine A-schemes $\operatorname{Spec} A[M \cap \sigma^\vee]$ where σ varies over the elements of Σ.

2.2 Proposition. *Let $\mathcal{F}$ be a quasi-coherent sheaf on X_Σ. Then we can compute the cohomology modules $H^k(X_\Sigma; \mathcal{F})$ as the cohomology of the* ČECH *cochain complex $\mathcal{C}^\bullet = (\mathcal{C}^\bullet, d)$ which is defined by*

$$\mathcal{C}^d = \bigoplus_{\substack{\sigma\in\Sigma \\ \operatorname{codim}\sigma = d}} \mathcal{F}^\sigma \ ,$$

with differential defined on direct summands by

$$\mathcal{F}^\sigma \xrightarrow{[\bar\sigma:\bar\tau]} \mathcal{F}^\tau \ .$$

Here $\mathcal{F}^\sigma$ denotes the module of sections of $\mathcal{F}$ over the affine open subset of X_Σ determined by the cone σ. In particular, $\mathcal{F}^\sigma$ is an $A[M \cap \sigma^\vee]$-module. The family $(\mathcal{F}^\sigma)_{\sigma\in\Sigma}$ of modules determines $\mathcal{F}$ completely.

This variant of ČECH cohomology should be well-known, but unfortunately there seems to be no published proof available. For the reader's convenience we give a proof in §4 below.

Torus-equivariant line bundles

2.3 We will apply Proposition 2.2 in the case where $\mathcal{F}$ is a torus equivariant algebraic line bundle on X_Σ. Recall [Oda88, §2] that such a sheaf is specified by a support function $h\colon N_\mathbb{R} \longrightarrow \mathbb{R}$ which is linear on each cone, and takes integral values on N; in other words, for each $\sigma \in \Sigma$ there exists $h_\sigma \in \hom_\mathbb{Z}(N, \mathbb{Z}) = M$ such that $h|_\sigma = (h_\sigma \otimes \mathrm{id}_\mathbb{R})\,|_\sigma$. The linear function h_σ is well-defined up to addition of a linear function which vanishes on $N \cap \sigma$.

2.4 The line bundle L_h corresponding to a support function h has a very explicit description: On the affine open set corresponding to $\sigma \in \Sigma$ the space of sections is the free $A[M \cap \sigma^\vee]$-module of rank 1 with basis $-h_\sigma$. Note that all these modules are contained in the free A-module $A[M]$ with basis M, hence we may consider them as M-graded A-modules.

2.5 We can apply Proposition 2.2 to the line bundle L_h. The resulting cochain complex of A-modules has a natural M-grading, and the differentials are homogeneous of degree 0 with respect to this grading (in the language of free modules, all

the terms in $\mathcal{C}^\bullet$ have a basis consisting of a subset of M, and all structure maps are induced by inclusion of subsets.) Hence the cohomology modules $H^k(X_\Sigma; L_h)$ have a direct sum decomposition

$$H^k(X_\Sigma; L_h) = \bigoplus_{\mathbf{b}\in M} H^k(X_\Sigma; L_h)_{\mathbf{b}} \ ,$$

with $H^k(X_\Sigma; L_h)_{\mathbf{b}}$ being isomorphic to the cohomology of the degree-$\mathbf{b}$ sub-cochain complex $\mathcal{C}^\bullet_{\mathbf{b}} = (\mathcal{C}^\bullet_{\mathbf{b}}, d)$ of $\mathcal{C}^\bullet$.

2.6 The cochain complex $\mathcal{C}^\bullet_{\mathbf{b}}$ itself admits a simple description: It is given by

$$\mathcal{C}^d_{\mathbf{b}} = \bigoplus_{\substack{\sigma\in\Sigma,\ \operatorname{codim}\sigma = d\\ \mathbf{b}+h_\sigma\in\sigma^\vee}} A$$

with differential induced by incidence numbers as before. Now if τ is a face of $\sigma \in \Sigma$ then $\sigma^\vee \subseteq \tau^\vee$, so $\mathbf{b} + h_\sigma \in \sigma^\vee$ implies $\mathbf{b} + h_\sigma \in \tau^\vee$, hence $\mathbf{b} + h_\tau \in \tau^\vee$ (for $h_\tau - h_\sigma \in \tau^\vee$, and $\tau^\vee$ is closed under addition since it is a convex cone). Thus the set

$$S(h, \mathbf{b}) := \bigcup_{\substack{\sigma\in\Sigma\\ \mathbf{b}+h_\sigma\in\sigma^\vee}} \bar\sigma \tag{2}$$

is a sub-complex of S^{n-1}, and $\mathcal{C}^\bullet_{\mathbf{b}}$ is nothing but the augmented cellular chain complex of $S(h, \mathbf{b})$, re-indexed suitably as a cochain complex. In other words, we have shown:

2.7 Theorem. *Suppose Σ is a complete fan in $N_\mathbb{R}$, and $h\colon N_\mathbb{R} \longrightarrow \mathbb{R}$ is a support function on Σ. Let L_h denote the algebraic line bundle on X_Σ associated to h (2.4), and define the space $S(h, \mathbf{b})$ as in* (2)*. For all $\mathbf{b} \in M$ there is an isomorphism of A-modules*

$$H^k(X_\Sigma; L_h)_{\mathbf{b}} \cong \tilde H_{n-1-k}(S(h, \mathbf{b}); A)$$

where $\tilde H_d(\,\cdot\,; A)$ denotes reduced cellular (or singular) homology with coefficients in A. □

For this to make sense, it is imperative to consider the augmented cellular chain complex to compute $\tilde H_d$ with augmentation concentrated in degree -1. In other words, $\tilde H_{-1}(\emptyset) = A$ by convention, while $\tilde H_{-1}(X) = 0$ whenever $X \neq \emptyset$.

The advantage of Theorem 2.7 over the standard result as given in [Oda88, Theorem 2.6] is that the former deals with the cell complex $S(h, \mathbf{b})$ arising as the intersection of a sub-fan of Σ with S^{n-1}, whereas the latter relies on computing certain subsets of $N_\mathbb{R}$ with a rather more delicate combinatorial structure.

The theorem leads immediately to some general observations. For example, the remark following Theorem 2.7 implies:

2.8 Corollary. *The top-dimensional cohomology is given by*

$$H^n(X_\Sigma; L_h)_{\mathbf{b}} = \begin{cases} 0 & \text{if there exists } \sigma \in \Sigma,\ \sigma \neq \{0\} \text{ with } \mathbf{b} + h_\sigma \in \sigma^\vee \\ A & \text{otherwise.} \end{cases}$$

Moreover, if $H^n(X_\Sigma; L_h)_{\mathbf{b}} = A$, then $H^k(X_\Sigma; L_h)_{\mathbf{b}} = 0$ for all $k \neq n$. □

Suppose now that K is a subcomplex of S^{n-1}. Then $\tilde{H}_{n-1}(K; A) \neq 0$ if and only if $K = S^{n-1}$. Indeed, if $K \neq S^{n-1}$ then K misses an $(n-1)$-dimensional cell of S^{n-1}, *i.e.*, there exists an n-dimensional cone $\sigma \in \Sigma$ such that K is contained in $S^{n-1} \setminus \operatorname{int} \bar{\sigma}$. Now $S^{n-1} \setminus \operatorname{int} \bar{\sigma}$ is contractible, hence has trivial reduced homology. The homology long exact sequence of the pair $(K, S^{n-1} \setminus \operatorname{int} \bar{\sigma})$ proves the assertion.

If there exists $\mathbf{b} \in M$ such that $\mathcal{S}(h, \mathbf{b}) = S^{n-1}$, then $\mathbf{b}$ is contained in the intersection of the closed half-spaces $-h_\rho + \rho^\vee$ where ρ varies over the 1-dimensional cones in Σ. Since Σ is complete, this implies that for all $\mathbf{a} \in \mathbb{Z}^n$ there exists $\rho \in \Sigma$ with $\mathbf{a} \in -h_\rho + \rho^\vee$, thus $\mathcal{S}(h, \mathbf{a}) \neq \emptyset$. Conversely, if $\mathcal{S}(h, \mathbf{b}) = \emptyset$ for some $\mathbf{b} \in M$ then there is no $\mathbf{a} \in M$ with $\mathcal{S}(h, \mathbf{a}) = S^{n-1}$ (in fact, there is a 1-dimensional cone $\rho \in \Sigma$ with $\mathbf{a} \notin -h_\rho + \rho^\vee$). Together with Theorem 2.7 this shows that the line bundle L_h cannot have global sections and nth cohomology and the same time:

2.9 Corollary. *At least one of the A-modules $H^0(X_\Sigma; L_h)$ and $H^n(X_\Sigma; L_h)$ is trivial.* □

On Example 1.3

Recall the notation from Example 1.3; we will use the field $A = \mathbb{C}$ of complex numbers. To work out the complex $\mathcal{S}(h, \mathbf{b}) \subseteq S^1$ for given $\mathbf{b} \in M$ one can start from a sketch of the halfspace arrangement $-h_{\rho_j} + \rho_j^\vee$, $j = 1, 2, 3, 4$ given by the shifted duals of the 1-dimensional cones in Σ. Furthermore, it is enough to consider those $\mathbf{b}$ which are contained in some bounded region of the resulting decomposition of $M_\mathbb{R}$ since $H^*(X_\Sigma; L_h)$ is finite-dimensional.

In our example, this leaves us to check contributions from five elements of M only. We will use coordinate notation for this paragraph. It is easily verified that $\mathcal{S}(h, (0, -1)^t) = \emptyset$, so $H^2(X_\Sigma; L_h)_{(0,-1)^t} = \tilde{H}_{-1}(\emptyset) = \mathbb{C}$. If $\mathbf{b}$ is one of the vectors $\{(0, 0)^t,\ (-1, 1)^t,\ (0, 1)^t,\ (1, 1)^t\}$, then $\mathcal{S}(h, \mathbf{b})$ is a 0-sphere corresponding to the intersection of the cones spanned by v_1 and v_3 with the unit sphere in $N_\mathbb{R} = \mathbb{R}^2$. Thus $H^1(X_\Sigma; L_h)_{\mathbf{b}} = \tilde{H}_0(S^0) = \mathbb{C}$ in these cases.

3 Proof of Theorem 1.2

The proof of Theorem 1.2 proceeds by verifying a LAURENT series identity first. Let as before $h \colon N_\mathbb{R} \longrightarrow \mathbb{R}$ be a support function, and choose corresponding

linear functions $h_\sigma \in M$ for $\sigma \in \Sigma$ (2.3). Define a formal LAURENT power series

$$R[\Sigma, h] = \sum_{\sigma \in \Sigma} (-1)^{\operatorname{codim} \sigma} R[-h_\sigma + \sigma^\vee] \ . \tag{3}$$

Fix $\mathbf{a} \in \mathbb{Z}$; we want to consider the coefficient of $x^{\mathbf{a}}$ in $R[\Sigma, h]$. The summand corresponding to $\sigma \in \Sigma$ contributes 0 if $\mathbf{a} + h_\sigma \notin \sigma^\vee$, and it contributes $(-1)^{\operatorname{codim} \sigma}$ otherwise. Since $0^\vee = \mathbb{R}^n$ we get a contribution of $(-1)^n$ for $\sigma = 0$ always. In other words, the coefficient of $x^{\mathbf{a}}$ is the EULER characteristic of the chain complex $\mathcal{C}^\bullet_{\mathbf{a}}$ (2.6), using $A = \mathbb{C}$ again:

$$\sum_{k=0}^{n} (-1)^k \dim_{\mathbb{C}} \mathcal{C}^k_{\mathbf{a}} = \chi(\mathcal{C}^\bullet_{\mathbf{a}})$$

The EULER characteristic can be computed using the cohomology groups of the cochain complex as well, so the coefficient of $x^{\mathbf{a}}$ is given by

$$\chi(\mathcal{C}^\bullet_{\mathbf{a}}) = \sum_{k=0}^{n} (-1)^k \dim_{\mathbb{C}} \tilde{H}_{n-1-k} S(h, \mathbf{a}) \ . \tag{4}$$

Using Theorem 2.7 we see that this is equal to $\sum_{k=0}^{n} (-1)^k \dim_{\mathbb{C}} H^k(X_\Sigma; L_h)_{\mathbf{a}}$. Since the cohomology of L_h is finitely generated (the variety X_Σ is complete by hypothesis), we see that this coefficient is zero for almost all $\mathbf{a} \in \mathbb{Z}^n$. In particular, $R[\Sigma, h]$ is a LAURENT polynomial.

Let Π denote the P-submodule of S generated by the rational functions corresponding to rational polyhedral cones. According to [Ish90, Theorem 1.2] there is a unique P-linear homomorphism (here $Q(P)$ denotes the quotient field of P as before)

$$\rho \colon \Pi \longrightarrow Q(P)$$

with $\rho(R[b + \sigma]) = R(b + \sigma)$ for all $b \in M_{\mathbb{R}}$ and all pointed rational polyhedral cones $\sigma \subset M_{\mathbb{R}}$ (see also [BHS, Theorem 2.4]). Note that ρ preserves LAURENT polynomials as they are finite sums of LAURENT power series associated to sets of the form $\mathbf{a} + \{0\}$. In particular, $\rho(x^{\mathbf{b}}) = x^{\mathbf{b}} \in P \subset Q(P)$ for all $\mathbf{b} \in M$.—If the rational polyhedral cone σ contains a line, then it can be shown that $\rho(K[\sigma]) = 0$, cf. [Ish90, Lemma 2.1] or [BHS, Lemma 2.5].

We now apply the homomorphism ρ to the LAURENT power series $R[\Sigma, h]$. On the one hand, we have

$$\begin{aligned}
\rho(R[\Sigma, h]) &= \sum_{\sigma \in \Sigma} (-1)^{\operatorname{codim} \sigma} \rho(R[-h_\sigma + \sigma^\vee]) \\
&= \sum_{\substack{\sigma \in \Sigma \\ \dim \sigma = n}} (-1)^{\operatorname{codim} \sigma} \rho(R[-h_\sigma + \sigma^\vee]) \\
&= \sum_{\substack{\sigma \in \Sigma \\ \dim \sigma = n}} R(-h_\sigma + \sigma^\vee) \\
&= R(\Sigma, h) \ .
\end{aligned}$$

(The second equality comes from the fact that if $\operatorname{codim}\sigma > 0$, then the dual cone $\sigma^\vee$ contains a line.) On the other hand, we have seen that $R[\Sigma, h]$ is a LAURENT polynomial. Hence $R(\Sigma, h) = \rho(R[\Sigma, h]) = R[\Sigma, h]$ is a LAURENT polynomial as well, and as seen before the coefficient of $x^{\mathbf{a}}$ is given by $\chi(H^*(X_\Sigma; L_h)_{\mathbf{a}})$. This finishes the proof.

As a final remark, we can also use Equation (4) to identify the coefficients of the monomials in $R(\Sigma, h)$ as this is an intermediate step in the above proof. The result then reads:

3.1 Corollary. *The coefficient of $x^{\mathbf{a}}$ in $R(\Sigma, h)$ is equal to $(-1)^{n-1}\tilde{\chi}(S(h,\mathbf{a}))$, the reduced* EULER *characteristic of the cell complex $S(h,\mathbf{a})$. In other words,*

$$R(\Sigma, h) = (-1)^{n-1} \sum_{\mathbf{a}\in\mathbb{Z}^n} \tilde{\chi}(S(h,\mathbf{a})) \cdot x^{\mathbf{a}} \ . \qquad \square$$

4 Proof of Proposition 2.2

Let A denote a commutative ring with unit. For a complete fan Σ in $N_{\mathbb{R}}$ we let X_Σ denote the associated toric scheme defined over A. A quasi-coherent sheaf of modules $\mathcal{F}$ on X_Σ determines, by evaluation on affine pieces, a diagram of A-modules

$$D(\mathcal{F}) = D\colon \Sigma^{\mathrm{op}} \longrightarrow A\text{--Mod}, \quad \sigma \mapsto D^\sigma = \mathcal{F}^\sigma$$

(where as before $\mathcal{F}^\sigma$ denotes the A-module of sections of $\mathcal{F}$ over the open affine subset of X_Σ determined by σ, cf. Proposition 2.2). The functor $\mathcal{F} \mapsto D(\mathcal{F})$ is exact: A short exact sequence of quasi-coherent sheaves yields a short exact sequence of diagrams. We need the fact that we can compute sheaf cohomolgy by higher derived limits of the associated diagram:

4.1 Lemma. *There are canonical isomorphisms*

$$H^j(X_\Sigma; \mathcal{F}) \cong \varprojlim{}^j D(\mathcal{F}) \ .$$

Proof. Given a cone $\sigma \in \Sigma$ write U_σ for the open affine subset of X_Σ determined by σ. Then by construction $D(\mathcal{F})^\sigma = \Gamma(U_\sigma; \mathcal{F})$, and the case $j = 0$ of the Lemma is just the sheaf axiom: A global section is uniquely determined by a collection of compatible local section.

Recall now that sheaf cohomology can be computed with flasque resolutions. That is, considering $\mathcal{F}$ as a sheaf of ABELian groups, choose a resolution

$$\mathcal{F} \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G}_1 \longrightarrow \cdots \tag{5}$$

with all the $\mathcal{G}_i$ being flasque. Let $U \subseteq X_\Sigma$ be an open subset; then $\mathcal{G}_i|_U$ is flasque, and $H^j(U; \mathcal{F})$ is isomorphic to the cohomology groups of the cochain complex

$$\Gamma(U; \mathcal{G}_0|_U) \longrightarrow \Gamma(U; \mathcal{G}_1|_U) \longrightarrow \cdots \ . \tag{6}$$

Passing to associated diagrams of ABELian groups the resolution (5) gives rise to a cochain complex

$$D(\mathcal{F}) \longrightarrow D(\mathcal{G}_0) \longrightarrow D(\mathcal{G}_1) \longrightarrow \cdots . \tag{7}$$

We claim that this is in fact a resolution of $D(\mathcal{F})$, considered as a diagram of ABELian groups. Indeed, given a cone $\sigma \in \Sigma$ the cochain complex

$$D(\mathcal{G}_0)^\sigma \longrightarrow D(\mathcal{G}_1)^\sigma \longrightarrow \cdots$$

is nothing but the cochain complex (6) for $U = U_\sigma$, hence its jth cohomology group is isomorphic to $H^j(U_\sigma : \mathcal{F})$. But U_σ is affine and $\mathcal{F}$ quasi-coherent, so these groups vanish for $j \geq 1$, proving the claim.

We observe that the resolution (7) is flasque in the sense that the canonical restriction maps

$$D(\mathcal{G}_i)^\sigma \longrightarrow \lim_{\tau \subset \sigma} D(\mathcal{G}_i)^\tau \tag{8}$$

are surjective. Indeed, using the definition of associated diagrams, the map (8) corresponds to the restriction map

$$\Gamma(U_\sigma; \mathcal{G}_0) \longrightarrow \Gamma(\bigcup_{\tau \subset \sigma} U_\tau; \mathcal{G}_i)$$

which is surjective since $\mathcal{G}_i$ is flasque. Hence we can use the resolution (7) to compute higher derived inverse limuits of $D(\mathcal{F})$ by applying the functor lim to (7), then taking cohomology groups. However, applying lim to (7) yields precisely the cochain complex (6) for $U = X_\Sigma$, which computes $H^j(X_\Sigma; \mathcal{F})$. Taking into account the well-known fact that the higher derived inverse limits of a diagram of A-modules can be computed in the category of diagrams of ABELian groups, we have thus proved the Lemma. □

The proof of Proposition 2.2 thus reduces to proving the following claim:

4.2 Proposition. *Let $D \colon \Sigma^{\mathrm{op}} \longrightarrow A{-}\mathrm{Mod}$, $\sigma \mapsto D^\sigma$ be a diagram of A-modules, where A is an arbitrary ring with unit. Form the cochain complex $\mathcal{C}^\bullet = \mathcal{C}(D)^\bullet$ by setting*

$$\mathcal{C}^k = \mathcal{C}(D)^k = \bigoplus_{\substack{\sigma \in \Sigma \\ \operatorname{codim} \sigma = k}} D^\sigma \ ,$$

with differential defined on direct summands by

$$D^\sigma \xrightarrow{[\bar\sigma : \bar\tau]} D^\tau \ .$$

Then the ČECH *cohomology modules $\check{H}^k(D) = h^k(\mathcal{C}(D)^\bullet)$ are naturally isomorphic to the higher derived inverse limits $\lim^k(D)$.*

The proof of the proposition will occupy the rest of this section. First, for $n = 1$ we know that Σ consists of the zero-cone, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{\leq 0}$. The cochain complex $\mathcal{C}^\bullet$ has the form

$$D^{\mathbb{R}_{\geq 0}} \oplus D^{\mathbb{R}_{\leq 0}} \xrightarrow{+} D^{\{0\}} \ ,$$

and the result is well-known in this case.

We can thus restrict to the case $n \geq 2$. We extend the cell structure on S^{n-1} introduced in 2.1 to a regular cell structure on B^n with a single n-cell denoted $\bar{B}$. The canonical maps $\lim(D) \longrightarrow D^\sigma$, modified by the incidence numbers $[\bar{B} : \bar{\sigma}]$, assemble to a map

$$\iota\colon \lim(D) \longrightarrow \bigoplus_{\substack{\sigma\in\Sigma \\ \dim\sigma=n}} D^\sigma = \mathcal{C}^0$$

which is, by the properties of incidence numbers, a co-augmentation of the cochain complex $\mathcal{C}^\bullet$.

4.3 Lemma. *The map ι is injective and induces an isomorphism* $\lim(D) \cong \check{H}^0(D)$.

Proof. An element of $\lim(D)$ is determined by its images in the D^σ where σ ranges over all n-dimensional cones of Σ. Conversely, an element of $\mathcal{C}^0$ lies in the kernel of $\mathcal{C}^0 \longrightarrow \mathcal{C}^1$ if and only if its components in D^σ and D^τ agree in $D^{\sigma\cap\tau}$ where $\sigma, \tau \in \Sigma$ are n-dimensional cones with $(n-1)$-dimensional intersection. Such an element thus determines a unique element of $\lim D$ mapping to the given element of $\mathcal{C}^0$. □

Observe now that the functor $D \mapsto \check{H}^*(D)$ is a δ-functor [Wei94, §2.1]. Indeed, a short exact sequence of diagrams

$$0 \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow 0 \tag{9}$$

gives rise to a short exact sequence of cochain complexes, hence by the snake lemma to an associated natural long exact sequence in cohomology. Since $D \mapsto \lim^*(D)$ is a universal δ-functor [Wei94, §2.1 and §2.5], it follows that we have uniquely determined natural maps

$$\nu_k\colon \lim^k(D) \longrightarrow \check{H}^k(D)$$

such that ν_0 is the isomorphism of Lemma 4.3, and such that the ν_k give rise to a commutative ladder diagram in cohomology for every short exact sequence of the form (9).

To prove that the maps ν_k are isomorphisms, we consider a decreasing filtration of the diagram D. For $0 \leq j \leq n$ we write

$$\kappa_j D\colon \Sigma^{\mathrm{op}} \longrightarrow A\text{–Mod}, \quad \sigma \mapsto \begin{cases} D^\sigma & \text{if } \operatorname{codim}\sigma \leq j, \\ 0 & \text{else.} \end{cases}$$

4.4 Lemma. *The maps ν_k are isomorphisms for all diagrams of the form $\kappa_0 D$.*

Proof. The diagram $\kappa_0 D$ has non-zero values only on n-dimensional cones, hence $\check{H}^k(\kappa_0 D) = 0$ for $k > 0$. It is easy to check that $\lim^k(\kappa_0 D) = 0$ for $k > 0$ (for example, examine the cochain complex of [Wei94, Vista 3.5.12] which computes higher derived inverse limits). Thus $\nu_k \colon \lim^k(\kappa_0 D) \longrightarrow \check{H}^k(\kappa_0 D)$ is an isomorphism for all k. □

We proceed by induction on j and state the **induction hypothesis:** *The maps $\nu_k \colon \lim^k(\kappa_{j-1} D) \longrightarrow \check{H}^k(\kappa_{j-1} D)$ are isomorphisms for all $k \geq 0$ and all diagrams D.* The case $j = 1$ is covered by the previous Lemma.

We have a sequence of epimorphisms of diagrams

$$D = \kappa_n D \xrightarrow{e_n} \kappa_{n-1} D \xrightarrow{e_{n-1}} \ \dots \ \xrightarrow{e_1} \kappa_0 D$$

and consequently a collection of short exact sequences (for $1 \leq j \leq n$)

$$0 \longrightarrow \ker(e_j) \longrightarrow \kappa_j D \longrightarrow \kappa_{j-1} D \longrightarrow 0 \ .$$

Consider the associated ladder diagram for some fixed $k \geq 1$:

$$\begin{array}{ccccccccc}
\lim^{k-1}(\kappa_{j-1} D) & \longrightarrow & \lim^k(\ker e_j) & \longrightarrow & \lim^k(\kappa_j D) & \longrightarrow & \lim^k(\kappa_{j-1} D) & \longrightarrow & \lim^{k+1}(\ker e_j) \\
\downarrow{\scriptstyle \nu_{k-1}} & & \downarrow{\scriptstyle \nu_k} & & \downarrow{\scriptstyle \nu_k} & & \downarrow{\scriptstyle \nu_k} & & \downarrow{\scriptstyle \nu_{k+1}} \\
\check{H}^{k-1}(\kappa_{j-1} D) & \longrightarrow & \check{H}^k(\ker e_j) & \longrightarrow & \check{H}^k(\kappa_j D) & \longrightarrow & \check{H}^k(\kappa_{j-1} D) & \longrightarrow & \check{H}^{k+1}(\ker e_j)
\end{array}$$

By our induction hypothesis we know that the first and fourth vertical arrow are isomorphisms. In view of the 5-lemma it is enough to show that the second and fifth vertical arrow are isomorphisms as well. Since $\tau_n D = D$ this proves the assertion of Proposition 4.2.

We are left to show that the maps $\nu_k \colon \lim^k(\ker e_j) \longrightarrow \check{H}^k(\ker e_j)$ are isomorphisms for all k. Now the diagram $\ker e_j$ has non-trivial entries only on cones of codimension j, and can thus be written as a direct sum of "atomic" diagrams with a single non-trivial entry. Since both $\lim^k$ and $\check{H}^k$ commute with direct sums of atomic diagrams (the former for abstract reasons, the latter by direct inspection), the induction step is completed if we can verify the following assertion:

4.5 Lemma. *Let C be an A-module. Let $\tau \in \Sigma$ be a cone of codimension $j > 0$, and let C_τ denote the atomic Σ^{op}-diagram with non-trivial value C attained at τ. Then the maps $\nu_k \colon \lim^k(C_\tau) \longrightarrow \check{H}^k(C_\tau)$ are isomorphisms for all $k \geq 0$.*

(Note that by uniqueness of the natural maps ν_k the direct sum decomposition of the diagram $\ker e_j$ carries over to a direct sum decomposition of the corresponding ν_k.)

The Lemma will follow from a brute-force calculation. By construction, $\check{H}^k(C_\tau) = 0$ for $k \neq j$, and $\check{H}^j(C_\tau) = C$. We have $\lim(C_\tau) = 0$ since τ has positive codimension. To compute the higher derived inverse limits, we embed the diagram C_τ into a short exact sequence

$$0 \longrightarrow C_\tau \longrightarrow C_{\geq\tau} \longrightarrow C_{>\tau} \longrightarrow 0 \tag{10}$$

where we write

$$C_{\geq\tau}\colon \Sigma^{\mathrm{op}} \longrightarrow A\text{–Mod}, \quad \sigma \mapsto \begin{cases} C & \text{if } \sigma \supseteq \tau \ , \\ 0 & \text{else} \end{cases}$$

(with non-trivial structure maps identities), and the diagram $C_{>\tau}$ is defined similarly.

4.6 Lemma. *The higher derived inverse limits remain unchanged when the diagram $C_{\geq\tau}$ is restricted to the subcategory $\tau \downarrow \Sigma = \{\sigma \in \Sigma \,|\, \sigma \supseteq \tau\}$. More precisely, the canonical restriction maps $\lim_{\Sigma^{\mathrm{op}}}{}^k C_{\geq\tau} \longrightarrow \lim^k_{(\tau\downarrow\Sigma)^{\mathrm{op}}} (C_{\geq\tau}|_{(\tau\downarrow\Sigma)})$ are isomorphisms.*

Proof. This can be read off from the usual cochain complex computing higher derived inverse limits as given in [Wei94, Vista 3.5.12]. □

4.7 Lemma. *We have* $\lim(C_{\geq\tau}) = C$ *and* $\lim^k(C_{\geq\tau}) = 0$ *for* $k \geq 1$.

Proof. By Lemma 4.6 we get an isomorphism $\lim^k(C_{\geq\tau}) \cong H^k(N(\tau\downarrow\Sigma)^{\mathrm{op}}; C)$ for all $k \geq 0$, where N denotes the nerve of the category. But $\tau \downarrow \Sigma$ has an initial object, hence is contractible. □

4.8 Lemma. *We have* $\lim^k(C_\tau) = 0$ *for* $k \neq j$, *and* $\lim^j(C_\tau) = C$.

Proof. This follows from the long exact sequence associated to the short exact sequence (10) and the calculations in the previous two lemmas. □

As a consequence it is enough to consider the case $k = j$ in Lemma 4.5.

4.9 Lemma. *We have* $\check{H}^k(C_{\geq\tau}) = 0$ *for* $k > 0$.

Proof. In short, this follows from the fact that the cochain complex $\mathcal{C}(C_{\geq\tau})^\bullet$ is a re-indexed variant of the cellular chain complex computing the reduced homology of a $(j-1)$-sphere with coefficients in C, so $\check{H}^k(C_{\geq\tau}) \cong \tilde{H}_{j-1-k}(S^{j-1}; C)$.

In more detail, the poset $\tau \downarrow \Sigma$ is known to be isomorphic to the j-dimensional quotient fan Σ/τ as defined in [Oda88, Corollary 1.7]. It is a complete fan in the vector space $N_\mathbb{R}/\mathrm{span}(\tau)$, its cones are given by the images of $\sigma \in \tau \downarrow \Sigma$ under the quotient map $N_\mathbb{R} \longrightarrow N_\mathbb{R}/\mathrm{span}(\tau)$. The fan Σ/τ induces a cell structure on some unit sphere S^{j-1} in $N_\mathbb{R}/\mathrm{span}(\tau)$, and taking the incidence numbers coming from the fan Σ as defined before, we see that $\mathcal{C}(C_{\geq\tau})^\bullet$ is, up to re-indexing, an augmented cellular chain complex of S^{j-1}. This chain complex is slightly non-standard: The augmentation maps are given by id_C or $-\mathrm{id}_C$, depending on the

incidence numbers $[\bar{\sigma} : \bar{\tau}]$. However, it is not difficult to show that this chain complex is isomorphic to a standard chain complex for any choice of orientations of the cones in Σ/τ, the required isomorphism being constructed by induction of the dimension of the cones, starting with τ. We omit the details. $\square$

We are now ready to prove Lemma 4.5. Consider the following piece of the ladder diagram relating $\lim^k$ and $\check{H}^*$:

$$\begin{tikzcd} \lim^{j-1}(C_{\geq\tau}) \arrow[r] \arrow[d, "\cong"', "\nu_{j-1}"] & \lim^{j-1}(C_{>\tau}) \arrow[r, "f"] \arrow[d, "\cong"', "\nu_{j-1}"] & \lim^{j}(C_{\tau}) \arrow[r] \arrow[d, "\nu_j"] & \lim^{j}(C_{\geq\tau}) = 0 \\ \check{H}^{j-1}(C_{\geq\tau}) \arrow[r] & \check{H}^{j-1}(C_{>\tau}) \arrow[r, "g"] & \check{H}^{j}(C_{\tau}) \arrow[r] & \check{H}^{j}(C_{\geq\tau}) = 0 \end{tikzcd} \tag{11}$$

Both rows are exact. The entries on the right are trivial by Lemma 4.7 and direct inspection of the cochain complex $\mathcal{C}(C_{\geq\tau})^\bullet$, respectively. The first vertical map is an isomorphism; for $j = 1$ it is the map ν_0, and for $j > 1$ it follows from Lemmas 4.7 and 4.9. The second vertical map is an isomorphism in view of our induction hypothesis (note that $C_{>\tau} = \kappa_{j-1}C_{\geq\tau}$). From the Five Lemma we conclude that the third vertical map is an isomorphism as desired.

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